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# CHROMATIC NUMBERS OF CAYLEY GRAPHS ON $\mathbb{Z}$ AND RECURRENCE

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*Dedicated to the memory of Paul Erdős*

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In 1987 Paul Erdős asked me if the Cayley graph defined on $\mathbb{Z}$ by a lacunary sequence has necessarily a finite chromatic number. Below is my answer, delivered to him on the spot but never[1] published, and some additional remarks. The key is the interpretation of the question in terms of *return times of dynamical systems*.

## 1. The Cayley graph $\mathbb{Z}_\Lambda$

**1.1.** Let $\Lambda\subset\mathbb{N}$. By definition, the Cayley graph $\mathbb{Z}_\Lambda$ is the graph whose vertices are the integers, and whose edges are the pairs $\{(n,n+\lambda): n\in\mathbb{Z},\ \lambda\in\Lambda\}$.

The sequence $\Lambda=\{\lambda_j\}$ is *lacunary* (with parameter $\rho$) if $\lambda_{j+1}/\lambda_j\geq\rho>1$.

The *chromatic number* $\chi(\Lambda)=\chi(\mathbb{Z}_\Lambda)$ is the smallest number of colors needed to color $\mathbb{Z}_\Lambda$ such that vertices connected by an edge have different colors.

For $\tau\in\mathbb{T}$ we denote by $\|\tau\|$ the distance in $\mathbb{T}$ of $\tau$ to $0\in\mathbb{T}$.

**Theorem 1.1.** *If $\Lambda$ is lacunary then $\chi(\Lambda)<\infty$.*

The theorem is an immediate corollary of the following theorem:

[1] An account did appear recently in chapter 5 of [4].
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**Theorem 1.2.** *For every $\rho>1$ there exists an $\varepsilon=\varepsilon(\rho)>0$ such that for any lacunary $\Lambda$ with parameter $\rho$ there exist $\alpha\in\mathbb{T}$ such that $\|\lambda\alpha\|>\varepsilon$ for all $\lambda\in\Lambda$.*

**Proof of Theorem 1.1.** Given $\rho$, divide $\mathbb{T}$ into $M$ equal arcs $\{I_k\}$, with $M\varepsilon>1$. Using the $\alpha$ given by Theorem 1.2, set $C(n)=j$ if $n\alpha\in I_j$. ∎

**1.2.** The proof of Theorem 1.2 is immediate if $\rho$ is big, say $\rho\geq 5$. Set $\varepsilon=1/4$ and, given $\Lambda=\{\lambda_j\}$, write $A_j=\{\alpha\colon\|\lambda_j\alpha\|\geq 1/4\}$. The set $A_j$ is the union of $\lambda_j$ arcs of length $1/2\lambda_j$. Each component of $A_j$ contains two full components of $A_{j+1}$. It follows that $\cap_j A_j$ contains a Cantor set.

For $\rho$ close to 1 the proof needs to be adapted slightly, and we defer it to the following subsection. As I found out later, the question whether or not (a variation of) the statement of Theorem 1.2 is valid for all $\rho>1$ was raised by Erdős in [2] and answered independently in [1] and [3].

For the purpose of proving Theorem 1.1 we can get around this case by replacing the circle by the $d$-dimensional torus. Let $d$ be an integer big enough to guarantee $\rho^d\geq 5$. Write $\Lambda_k=\{\lambda_{k+jd}\}_{j=0}^{\infty}$, $k=1,\ldots,d$. Each $\Lambda_k$ is lacunary with ratio $\geq 5$ and we can choose $\alpha_k$ such that $\|\lambda\alpha_k\|\geq 1/4$ for $\lambda\in\Lambda_k$.

We now divide $\mathbb{T}^d$ into $5^d$ boxes $B_j$ by dividing the coordinate circles into five equal arcs each, and set $C(n)=j$ if $n(\alpha_1,\ldots,\alpha_d)\in B_j$. ∎

**1.3. Back to Theorem 1.2.** Given any sequence $\Lambda$ and any $L>1$, we denote $\Lambda_n=\Lambda\cap[L^n,L^{n+1}]$ and $m=m(\Lambda,L)=\sup_n\#\{\Lambda_n\}$. If $\Lambda=\{\lambda_j\}$ is *lacunary* with parameter $\rho>1$, then $\rho^m<L$ so that $m=m(\Lambda,L)<\infty$ for all $L$. Conversely, if $m(\Lambda,L)<\infty$ for some $L$ then $\Lambda$ is a finite union of lacunary sequences.

**Proof of Theorem 1.2.** We propose to prove a little more than is claimed in the theorem, namely the following statements:

**Claim 1.** *For every $\rho>1$ there exists*[2] *an $\varepsilon=\varepsilon(\rho)>0$, such that for any lacunary $\Lambda$ with parameter $\rho$ there exist $\alpha\in\mathbb{T}$ such that $\|\lambda\alpha\|>\varepsilon$ for all $\lambda\in\Lambda$.*

**Claim 2.** *If $\Lambda$ is a finite union of lacunary sequences then the set*

$$A(\Lambda)=\{\alpha\colon\exists\varepsilon\text{ such that for all }\lambda\in\Lambda\ \ \|\lambda\alpha\|>\varepsilon\}$$

*has Hausdorff dimension 1.*

[2] For $\rho$ close to 1 we have $\varepsilon(\rho)>(\rho-1)^2\log^{-2}(\rho-1)$.

Write $\rho = 1+1/p$; we focus now on the case $\rho$ close to 1, that is, large $p$. For the proof of Claim 1 start with an integer $L$ such that $L \approx 4p\log p$. Notice that, as $\rho^{(3/2)p\log p} \approx p^{3/2} >> L$, we have $m(\Lambda, L) < (3/2)p\log p \approx 3L/8$. If we replace $L$ by its square, $m$ only doubles; thus we can have the ratio $m/L$ as small as we want by replacing $L$ by a (large) power thereof.

Denote by $\mathcal{P}_k$ the partition of $\mathbb{T}$ by the roots of unity of order $L^{k+1}$. An atom $I_k$ of $\mathcal{P}_k$ is *$\Lambda$-proper* if for any $j < k$, any $\lambda \in \Lambda_j$, and any $t \in I_k$ we have $\|\lambda t\| > 1/2L^2$. Denote by $G_k$ the union of the $\Lambda$-proper atoms of $\mathcal{P}_k$.

We propose to show that every $I_k \in \mathcal{P}_k$ which is $\Lambda$-proper contains at least $L-4m$ $\Lambda$-proper elements of $\mathcal{P}_{k+1}$.

For $\lambda \in \Lambda_k$ write $B_\lambda = \{t \colon \|\lambda t\| < 1/2L^2\}$. The periodicity of $B_\lambda$ is $\lambda^{-1}$, bigger than $L^{-(k+1)}$, so that $B_\lambda \cap I_k$ consists of at most two arcs, each of length bounded by $\lambda^{-1}L^{-2} \le L^{-(k+2)}$, and hence can have nontrivial intersection with at most two $\mathcal{P}_{k+1}$ sub-intervals of $I_k$.

Now observe that any point in $\bigcap G_k$ answers Claim 1 with $\varepsilon = 1/2L^2$. Also, by taking larger $L$ the ratio $m/L \to 0$ and the dimension of $\bigcap G_k$ is as close to 1 as we want. Since $A(\Lambda)$ contains all of these, this proves Claim 2. ∎

## 2. Recurrence

**2.1.** What we have shown in the previous section is that a lacunary sequence *is not recurrent* (see the definition below) for translations on circles or higher dimensional tori. Instead of a translation on the circle or the $d$-dimensional torus we can use a *compact system* $(X, \rho, T)$, that is, a homeomorphism $T$ on a compact metric space $(X, \rho)$. The following theorem shows that this level of generality gives a complete characterisation of finite chromatic number.

**Theorem 2.1.** *$\chi(\Lambda) < \infty$ if, and only if, there exists a compact metric space $(X, \rho)$, a homeomorphism $T$ of $X$, and some $\varepsilon > 0$ such that for all $x \in X$ and $\lambda \in \Lambda$*

$$\rho(x, T^\lambda x) \ge \varepsilon\,.$$

**Proof.** The “if” part is exactly as above: partition $X$ into sets $B_j$ of diameter less than $\varepsilon$, choose $x_0 \in X$, and define $C(n) = j$ when $T^n x_0 \in B_j$.

For the “only if” part we do the following construction: Let $C \in \{1, \dots, d\}^{\mathbb{Z}}$ be a $d$-coloring of $\mathbb{Z}$, appropriate for $\mathbb{Z}_\Lambda$. Let $X_C$ be the closure in $\{1, \dots, d\}^{\mathbb{Z}}$ of the set of translates of $C$: $\{C_m(n) = C(m+n)\}_{m\in\mathbb{Z}}$. Remember that the metric in $\{1, \dots, d\}^{\mathbb{Z}}$ is defined by

$$\rho(\{a_n\}, \{b_n\}) = 1/(1 + \max\{k : |j| \le k \Rightarrow a_j = b_j\}).$$

Define $T$ to be the translation: $T(\{a_n\})=\{a_{n+1}\}$.

For $x\in X_C$ and any $\lambda\in\Lambda$, $x_\lambda=(T^\lambda x)_0\neq x_0$, and $\rho(x,T^\lambda x)=1$. ∎

Recall that the system $(X,\rho,T)$ is *minimal* if any nonempty, $T$-invariant, closed subset of $X$ coincides with $X$. Conditions equivalent to minimality are:

**a.** The $T$-orbit of any $x\in X$ is dense in $X$.

**b.** If $O\subset X$ is open, then $\cup T^n O = X$. Since $X$ is assumed compact, a finite number of $T^n O$'s is sufficient to cover $X$.

**Definition 1.** A sequence $\Lambda\subset\mathbb{N}$ is *recurrent* for a system $(X,\rho,T)$ if for any open set $O\subset X$ there exist $\lambda\in\Lambda$ such that $O\cap T^{-\lambda}O\neq\emptyset$.

$\Lambda$ is *topologically recurrent* if it is recurrent for every minimal topological system. We denote the class of all topologically recurrent $\Lambda$'s by $\mathcal{TR}$.

**Remark.** The definitions of minimality and recurrence make no use of the metric $\rho$ but for the fact that it defines the topology. If we have a homeomorphism $T$ of a compact space $X$, and a non-recurrent sequence $\Lambda$, i.e., an open $O\subset X$ such that $O\cap T^{-\lambda}O=\emptyset$ for all $\lambda\in\Lambda$, we obtain as above that $\chi(\Lambda)<\infty$ and by Theorem 2.1 there exists a metric system[3] for which $\Lambda$ is non-recurrent. It follows that checking recurrence only on *metric* systems is equivalent to checking it for all compact systems.

Theorem 2.1 says that $\chi(\Lambda)=\infty$ if, and only if, $\Lambda\in\mathcal{TR}$.

**2.2. Bohr recurrence.** Translations on circles or higher dimensional tori are not just homeomorphisms, they are isometries. Conversely, a minimal topological system $(X,\rho,T)$ in which $T$ is an isometry[4], is isomorphic[5] to a translation on a compact monothetic group, that is, a group that has a dense subgroup isomorphic to $\mathbb{Z}$.

If $\mathbb{G}$ is a compact monothetic group and $g_0$ generates a dense subgroup, every character $\gamma$ on $\mathbb{G}$ is completely determined by the value $\langle\gamma,g_0\rangle$. The dual $\hat{\mathbb{G}}$ can therefore be identified with the set of $\langle\gamma,g_0\rangle$, which is a subgroup of $\mathbb{T}_d$, the circle[6] endowed with the discrete topology.

Monothetic groups can thus be characterized as duals of subgroups of $\mathbb{T}_d$. Equivalently they are factor groups of $\mathbb{B}$, the Bohr compactification of $\mathbb{Z}$, which is the dual group of the entire $\mathbb{T}_d$.

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[3] It is the factor system obtained by taking the topology spanned by $\{T^n O\}$.

[4] It suffices to assume that $\{T^n\}_{n\in\mathbb{Z}}$ is equicontinuous, since then $T$ is an isometry for a metric $\rho'$ equivalent to $\rho$.

[5] Here is the idea of the proof. Define *addition* on $X$ as follows: take an arbitrary $x_0\in X$, denote $x_n=T^n x_0$, define $x_n$"+"$x_m=x_{n+m}$ and extend the operation to $X\times X$ by continuity.

[6] We identify implicitly the group of complex numbers of modulus 1 with $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

**Definition 2.** $\Lambda$ is *Bohr recurrent*, denoted $\Lambda \in \mathcal{BR}$, if it is recurrent for every translation on the Bohr group $\mathbb{B}$.

Since recurrence for a system implies (trivially) recurrence for any factor thereof, Bohr recurrence is equivalent to recurrence for all minimal *isometries.*

**2.3.** The recurrence properties, or chromatic number, of a sequence $\Lambda$ depend on its *arithmetic richness* rather than its size. For example: $\chi(2\mathbb{Z})=\infty$ while $\chi(2\mathbb{Z}+1)=2$. A striking albeit trivial example of the effect of arithmetic structure rather than size is: let $E\subset\mathbb{Z}$ be infinite, and $\Lambda=E-E=\{e_1-e_2: e_j\in E\}$. Then $\chi(\Lambda)=\infty$.

The "arithmetic richness" needed for Bohr recurrence is given by the following lemma.

**Lemma 2.1.** *$\Lambda\in\mathcal{BR}$ if, and only if, for any finite subset $\{\alpha_j\}_1^N\subset\mathbb{T}$ and any $\varepsilon>0$ there exist $\lambda\in\Lambda$ such that $\|\lambda\alpha_j\|<\varepsilon$ for every $j\in[1,\dots,N]$.*

**Proof.** Consider the translation by $g_0\in\mathbb{B}$. $\Lambda$ is recurrent for this translation if (and only if) given any neighborhood $U$ of the identity in $\mathbb{B}$, there exists $\lambda\in\Lambda$ such that $\lambda g_0\in U$.

A basis of the topology of $\mathbb{B}$ at the identity consists of sets defined as follows: $V=\{g:|\langle\gamma_j,g\rangle-1|<\varepsilon\}$ where $\{\gamma_j\}$ is a finite set of characters and $\varepsilon>0$. Define $\alpha_j$ by: $e^{2\pi i\alpha_j}=\langle\gamma_j,g_0\rangle$. The condition $|\langle\gamma_j,\lambda g_0\rangle-1|<\varepsilon$ is equivalent to $\|\lambda\alpha_j\|<\varepsilon$. ∎

**Definition 3.** $\Lambda$ is *Bohr$(n)$ recurrent* if it is recurrent for every translation of $\mathbb{T}^n$. We denote the class of all such $\Lambda$'s by $\mathcal{BR}(n)$.

The criterion for $\mathcal{BR}(n)$ is the same as that of Lemma 2.1 except that the size of the set $\{\alpha_j\}$ is limited to $n$.

**2.4.** The conditions that define $\mathcal{BR}(n)$ get stricter as $n$ increases; it is clear that $\mathcal{BR}(n)\subset\mathcal{BR}(n-1)$ and we are about to see that the inclusion is proper.

**Lemma 2.2.** *Assume $\Lambda\in\mathcal{BR}(n-1)$. If $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{T}^n$ is a generator, then the set $\overline{\{\lambda\alpha:\lambda\in\Lambda\}}$ intersects every closed infinite subgroup of $\mathbb{T}^n$.*

*Conversely, if $V\subset\mathbb{T}^n$ is open and intersects every closed infinite subgroup of $\mathbb{T}^n$ and if $\alpha$ is a generator of $\mathbb{T}^n$, then*

$$\Lambda=\{\lambda:\lambda\alpha\in V\} \tag{2.1}$$

*is in $\mathcal{BR}(n-1)$.*

**Proof.** Assume $\Lambda\in\mathcal{BR}(n-1)$, and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{T}^n$ a generator. Let $G$ be an infinite closed subgroup[7] of $\mathbb{T}^n$, and let $G^\perp$ be the subgroup of $\mathbb{Z}^n$ which is trivial on $G$ (its "annihilator"). Let $\mathbf{b}_k=(b_{k,1},\dots,b_{k,n})$, $k=1,\dots,m<n$, be a basis for $G^\perp$, so that $G=\{\tau:\langle\mathbf{b}_k,\tau\rangle=1, 1\le k\le m\}$.

Write $\beta_k=\sum b_{k,j}\alpha_j$, $k=1,\dots,m$. Since $\Lambda\in\mathcal{BR}(n-1)$, given $\varepsilon>0$, there exist $\lambda\in\Lambda$ such that $\|\lambda\beta_k\|<\varepsilon$ for all $k$, which means $\|\sum b_{k,j}\lambda\alpha_j\|<\varepsilon$, i.e., $|\langle\mathbf{b}_k,\lambda\alpha\rangle-1|<\varepsilon$ for all $k$, and $\lambda\alpha$ is $\varepsilon$-close to $G$. This being true for all $\varepsilon>0$, we obtain $\overline{\{\lambda\alpha:\lambda\in\Lambda\}}\cap G\neq\emptyset$.

Conversely, let $V\subset\mathbb{T}^n$ and $\alpha=(\alpha_1,\dots,\alpha_n)$ be as described in the statement, and let $\Lambda$ be defined by (2.1).

Let $\{t_k\}_1^{n-1}\subset\mathbb{T}$. Choose a maximal subset of the $t$'s such that these, along with $\{\alpha_j\}\cup\{1\}$, are rationally independent. Rearranging the $t$'s, we may assume that (for some integer $s\in[0,n-1]$) $\{t_j\}_{j=s+1}^{n-1}\cup\{\alpha_j\}\cup\{1\}$ is rationally independent, while each $t_j$, $j=1,\dots,s$, is a rational combination of these. With an appropriate (common denominator) $N$ we have for $j=1,\dots,s$, and integer coefficients $a_{j,k}$, $b_{j,l}$, $N_j$,

$$Nt_j=\sum_{k=1}^{n}a_{j,k}\alpha_k+\sum_{l=s+1}^{n-1}b_{j,l}t_l+N_j. \tag{2.2}$$

Denote $\mathbf{a}_j=(a_{j,1},\dots,a_{j,n})$, let $D\subset\mathbb{Z}^n$ be the subgroup generated by $\{\mathbf{a}_j\}_{j=1}^{s}$, and $D^\perp\subset\mathbb{T}^n$ its annihilator. The connected component $D_0^\perp$ of the identity in $D^\perp$ is closed and non-trivial, and by our assumption has a non empty intersection with $V$. Choose a point $(c_1,\dots,c_n)\in V\cap D_0^\perp$; let $(c_1^*,\dots,c_n^*)\in D_0^\perp$ be such that $N(c_1^*,\dots,c_n^*)=(c_1,\dots,c_n)$.

The independence of $\{t_j\}_{j=s+1}^{n-1}\cup\{\alpha_j\}\cup\{1\}$ implies that for every $\varepsilon_1>0$ there exist integers $\lambda_1$ such that

$$\|\lambda_1 t_j\|<\varepsilon_1,\quad s+1\le j\le n-1 \text{ and } \left\|\lambda_1\alpha_j-c_j^*\right\|<\varepsilon_1,\quad 1\le j\le n. \tag{2.3}$$

Given $\varepsilon>0$, if $\varepsilon_1$ is sufficiently small, then (2.3) implies that $\lambda=N\lambda_1\in\Lambda$ and

$$\|\lambda t_k\|<\varepsilon,\quad 1\le k<n. \tag{2.4}$$

∎

The sequences defined by (2.1) are clearly *not* in $\mathcal{BR}(n)$ when $V$ is at a positive distance from the identity. Any sequence $\Lambda\in\mathcal{BR}(n-1)\setminus\mathcal{BR}(n)$ is contained in one defined by (2.1) for an appropriate $V$ and $\alpha$. In fact if the $\mathcal{BR}(n)$ condition fails for $\alpha$, one can take for $V$ any neighborhood of $\Lambda\alpha$ which stays away from the identity.

[7] hence of positive dimension

Thus[8], if $\alpha_1,\dots,\alpha_n$ are rationally independent mod 1, and we define

$$\Lambda = \{\lambda : \min_j \|\lambda\alpha_j - 1/2\| < .01\} \tag{2.5}$$

then $\Lambda \in \mathcal{BR}(n-1)\setminus\mathcal{BR}(n)$.

The following lemma is a restatement of Lemma 2.2 for $\Lambda$-orbits in $\mathbb{T}^{n+N}$ when $\Lambda\in\mathcal{BR}(n)$. The proof above applies essentially verbatim,

**Lemma 2.3.** *Assume $\Lambda\in\mathcal{BR}(n)$. If $\alpha=(\alpha_1,\dots,\alpha_{n+N})\in\mathbb{T}^{n+N}$ is a generator, then the set $\overline{\{\lambda\alpha:\lambda\in\Lambda\}}$ intersects every closed $N$-dimensional subgroup of $\mathbb{T}^{n+N}$.*

*Conversely, if $V\subset\mathbb{T}^{n+N}$ is open and intersects every closed $N$-dimensional subgroup of $\mathbb{T}^{n+N}$ and if $\alpha$ is a generator of $\mathbb{T}^{n+N}$, then*

$$\Lambda = \{\lambda : \lambda\alpha \in V\} \tag{2.6}$$

*is in $\mathcal{BR}(n)$.*

## 3. Universal sequences

For $\Lambda\subset\mathbb{N}$ write $\Lambda_{[1,N]}=\Lambda\cap[1,\dots,N]$.

**Lemma 3.1.** $\chi(\Lambda)=\sup_{N\to\infty}\chi(\Lambda_{[1,N]})$.

**Proof.** Clearly $\chi(\Lambda)\geq\sup_{N\to\infty}\chi(\Lambda_{[1,N]})$. Assume $\chi(\Lambda_{[1,N]})\leq k$ for all $N$. Let $c_N\in\{1,\dots,k\}^{\mathbb{Z}}$ be an appropiate coloring for $\Lambda_{[1,N]}$ and let $C$ be a limit point of $c_N$ in $\{1,\dots,k\}^{\mathbb{Z}}$. $C$ is an appropiate coloring for $\Lambda$. ∎

**Proposition 3.1.** *Let $\Lambda_j\subset[1,L_j]$ be finite sequences of positive integers, with uniformly bounded chromatic number, say $\chi(\Lambda_j)\leq k$. Let $m_j\geq 5L_j$, and $M_j=\prod_{l<j}m_l$. Write $\Lambda=\cup_{j=1}^{\infty}M_j\Lambda_j$. Then $\chi(\Lambda)\leq k^2$.*

**Proof.** By Lemma 3.1 it suffices to show that $\chi(\cup_{j=1}^{N}M_j\Lambda_j)\leq k^2$ for all $N$. In what follows we take as colors the elements of the cyclic group $\mathbb{Z}_k=\mathbb{Z}/k\mathbb{Z}$.

We express any integer $n\in\mathbb{Z}$ (uniquely) in the form, referred to as *the standard $N$-expansion*, $\sum_{j=1}^{N+1}a_jM_j$ where $0\leq a_j<m_j$ for $j=1,\dots,N$ and $a_{N+1}\in\mathbb{Z}$. Let $c_j\in\mathbb{Z}_k^{\mathbb{Z}}$ be an appropriate coloring for $\Lambda_j$, let $w_j$ be a word of length $m_j$ in $c_j$, $j=1,\dots,N$, and write $\tilde{C}(n)=\sum_{j=1}^{N}w_j(a_j)$.

[8] The examples given by 2.5 are not only in $\mathcal{BR}(n-1)$, they have the stronger property that for any $\beta=(\beta_1,\dots,\beta_{n-1})$, the set $\{\lambda\beta:\lambda\in\Lambda\}$ is dense in $\mathbb{T}^{n-1}$.

$\tilde{C}$ is "almost" an appropriate coloring for $\cup_{j=1}^{N} M_j \Lambda_j$; if $n_1$ and $n_2$ have all but one, say $j = s$, of the first $N$ digits equal, and if $n_2 - n_1 \in M_s \Lambda_s$, then $\tilde{C}(n_2) \neq \tilde{C}(n_1)$ since $w_s$ assumes distinct values on the corresponding coefficients while all the other readings are identical. In general, if $\lambda \in \Lambda_r$ and if the coefficient $a_r$ of $n$ is smaller than $.8m_r < m_r - L_r$, then $n$ and $n + M_r\lambda$ do have all the coefficients but $a_r$ equal and by the previous remark $\tilde{C}$ assumes different values on the two. If we set $I_N = \sum_{j=2}^{N} i_j M_j$ with $i_j \approx \frac{1}{2} m_j$, then for any $n$ and any $\lambda \in \cup_{j=1}^{N} M_j \Lambda_j$ one at least of the pairs $n, n+\lambda$ and $n + I_N, n + I_N + \lambda$ fits the description above and it follows that $C(n) = (\tilde{C}(n), \tilde{C}(n + I_N))$ is an appropriate coloring, with no more than $k^2$ colors, for $\cup_{j=1}^{N} M_j \Lambda_j$. ∎

**Definition 1.** If $\Lambda$ and $\Lambda_1$ are sequences of positive integers, we say that $\Lambda$ *scale-contains* $\Lambda_1$ if for some integer $m$, $\Lambda \supset m\Lambda_1$.

We can take as $\{\Lambda_j\}$ an arrangement of all the finite sequences of chromatic number bounded by $k$; the proposition above implies:

**Theorem 3.1.** *Given $k \in \mathbb{N}$, there exist a sequence $\Lambda$ of chromatic number bounded by $k^2$ which is $k$-*universal*, i.e. scale-contains every finite sequence of chromatic number bounded by $k$.*

## 4. Is $\mathcal{BR} = \mathcal{TR}$?

Theorem 1.1 is equivalent to the statement that lacunary sequences are not topologically recurrent, and we proved more, namely that they are not even in $\mathcal{BR}(1)$.

Denote $\tilde{\chi}(n) = \inf \chi(\Lambda), \Lambda \in \mathcal{BR}(n)$.

**Theorem 4.1.** *The statement $\mathcal{BR} \neq \mathcal{TR}$ is equivalent to $\tilde{\chi}(n) = O(1)$.*

**Proof.** Assume $\tilde{\chi}(n) = O(1)$. Let $\Lambda_n \in \mathcal{BR}(n)$ with $\chi(\Lambda_n) < 2\tilde{\chi}(n)$. Let $\widetilde{\Lambda}_n$ be $\Lambda_n \cap [1, N_n]$ with $N_n$ large enough to guarantee that for for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$ the distance of $\widetilde{\Lambda}_n(\alpha_1, \ldots, \alpha_n)$ to 0 is less than $1/n$. Notice that the condition remains valid if we replace $\widetilde{\Lambda}_n$ by any integer multiple thereof. Now apply Proposition 3.1; the sequence obtained is in $\mathcal{BR} \setminus \mathcal{TR}$.

The opposite implication is obvious. ∎

**Remark.** The question can be stated in terms of finite sets only, as is apparent from the proof above. If we denote by $\widetilde{\mathcal{BR}}(n)$ the set of all *finite* sequences $\Lambda$ such that for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$ the distance of $\Lambda(\alpha_1, \ldots, \alpha_n)$ to 0 is less than $1/n$, and set

$$\chi^*(n) = \inf_{\Lambda \in \widetilde{\mathcal{BR}}(n)} \chi(\Lambda)$$

then the boundedness of $\chi^*(n)$ is equivalent to $\mathcal{BR} \neq \mathcal{TR}$.

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